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Approximation of infinite dimensional fractals generated by integral equations

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ABSTRACT

We present some results concerning fractals generated by an iterated function system in the infinite dimensional space of continuous functions on a compact interval. Namely, we approximate the fractal via a finite approximant set and project this approximant set in two dimensions, in order to "draw" a picture of it.

1. Introduction

Iterated function systems (IFS) were introduced in their present form by John Hutchinson in [1] and popularized by Michael Barnsley in [2]. They provide a convenient way to describe and classify deterministic fractals in the form of a recursive definition.

In recent years, IFSs have attracted much attention; they were used by researchers working on autoregressive time series, engineering sciences, physics, etc. For applications of IFSs in image processing theory, in the theory of stochastic growth models and in the theory of random dynamical systems one can consult [3–5].

On the one hand, since the appearance of Hutchinson's paper, many papers containing several types of generalizations of the iterated function systems theory appeared. Actually there is a current effort to extend Hutchinson's classical framework for fractals to more general spaces and infinite IFSs. For example, G. Gwóźdź-Łukowka and J. Jakymski discuss the Hutchinson–Barnsley theory for infinite iterated function systems in [6]; A. Łoziński, K. Źyczkowsi and W. Słomczyński introduce the notion of quantum iterated function systems (QIFS) which is designed to describe certain problems of nonunitary quantum dynamics in [7]; A. Käenmäki constructs a thermodynamical formalism for very general iterated function systems in [8]; K. Leśniak presents a multivalued approach of infinite iterated function systems in [9].

On the other hand, the problem of fractals' approximation is extremely important from the practical point of view (see for example [10] for practical applications on astronomy, [11] for image processing, [12] for surface modeling, shape description and geometric surface compression). Numerical comparisons among approximations of a fractal set are presented in [13]. New algorithms for approximation of fractal sets can be found in [14,15].

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The aim of this paper is to present some results concerning fractals generated by an iterated function system in the infinite dimensional space of continuous functions on a compact interval. Namely, we approximate the fractal via a finite approximant set and project this approximant set in two dimensions, in order to "draw" a picture of it.

2. Preliminaries

Let $a < b$ be real numbers. In the sequel X (respectively Y) will be the Banach space of all continuous $f : [a, b] \to \mathbb{R}$ (respectively $K : [a, b] \times [a, b] \to \mathbb{R}$) equipped with the sup norm.

Let $K^1, K^2, \ldots, K^h$ be in Y and $C = \max_{i=1}^{h} \|K^i\|$.

Let $\lambda \in \mathbb{R}$ be such that

$$|\lambda| (b-a)C < 1, \tag{1}$$

and let $\varepsilon > 0$ be such that

$$r = |\lambda| (b-a)(C+\varepsilon) < 1. \tag{2}$$

Also, take $f^1, f^2, \ldots, f^h$ in X and put

$$M = \max_{i=1}^{h} \|f^i\| . \tag{3}$$

Now, it is possible to define the contractions $T^i : X \to X$, $i = 1, 2, \ldots, h$, given via $T^i(u) = v^i$, where

$$v^i(x) = f^i(x) + \lambda \int_a^b K^i(x, y)u(y)\mathrm{d}y,$$

because, taking into account (1), we have:

$$\|T^i(u_1) - T^i(u_2)\| = |\lambda| \left| \int_a^b K^i(x, y)(u_1(y) - u_2(y))\mathrm{d}y \right| \leq |\lambda| (b-a) \|K^i\| \|u_1 - u_2\|$$

and therefore all the T^i have contraction constants less than or equal to $|\lambda| (b-a)C$ which is strictly less than r.

The fixed point φ^i of T^i is the solution of the Fredholm integral equation

$$\varphi^i(x) = f^i(x) + \lambda \int_a^b K^i(x, y)\varphi^i(y)\mathrm{d}y.$$

The space Y includes the subspace S of all polynomial functions having the form $P(x, y) = \sum_{i,j} a_{i,j} x^i y^j$. It is known that S is dense in Y. Using this fact, we shall approximate the contractions T^i with other contractions T_n^i, as follows. For any $i = 1, 2, \ldots, h$, take a sequence $(P_n^i)_n$ in S such that $P_n^i \underset{n}{\to} K^i$. Namely, take the sequence $(P_n^i)_n$ such that

$$\|P_n^i - K^i\| \leq \varepsilon_n, \quad \text{for all } i = 1, 2, \ldots, h \tag{4}$$

where $0 < \varepsilon_n < \varepsilon$ and $\varepsilon_n \underset{n}{\to} 0$ (see (2)).

It follows that

$$\|P_n^i\| \leq \|K^i\| + \varepsilon_n \leq \|K^i\| + \varepsilon \leq C + \varepsilon \tag{4'}$$

and therefore all $T_n^i : X \to X$, $i = 1, 2, \ldots, h$, given via $T_n^i(u) = v_n^i$, where

$$v_n^i(x) = f^i(x) + \lambda \int_a^b P_n^i(x, y)u(y)\mathrm{d}y,$$

are contractions, having contraction constant less than or equal to $r = |\lambda| (b-a)(C+\varepsilon)$ (see (2) and (4′)).

It is easy to see that, for all n and i, one has, if $u \in X$:

$$\|T_n^i(u) - T^i(u)\| \leq r_n \|u\| , \tag{5}$$

where (see ((4)))

$$r_n = \max_{i=1}^{h} |\lambda| (b-a) \|P_n^i - K^i\| \leq |\lambda| (b-a)\varepsilon_n \underset{n}{\to} 0. \tag{6}$$

The fixed point of T_n^i is the solution of the Fredholm equation (with degenerate kernel P_n^i) of type

$$\varphi^i(x) = f^i(x) + \lambda \int_a^b \left(\sum_{s,t} a_{s,t}^i x^s y^t \right) \varphi^i(y) \mathrm{d}y$$

and has the form

$$\varphi^i(x) = f^i(x) + \sum_{t=0}^{m(i)} a_t^i x^t,$$

where $a_t^i \in \mathbb{R}$ are uniquely determined.

3. Approximation results for fractals generated by Fredholm integral equations

Let $\mathcal{K}(X)$ be the set of all non empty compact subsets of X, which becomes a complete metric space when equipped with the Hausdorff–Pompeiu metric H, given via

$$H(A, B) = \max(d(A, B), d(B, A)),$$

where

$$d(A, B) = \sup\{dist(u, B) \mid u \in A\}$$

and

$$dist(u, B) = \inf\{\|u - b\| \mid b \in B\}.$$

In particular, we have $dist(u, \{x_0\}) = \|u - x_0\|$.

The Hausdorff–Pompeiu metric can be defined alternatively as follows. For any $r > 0$ and any $M \in \mathcal{K}(X)$ put

$$B(M, r) = \{x \in X \mid dist(x, M) < r\}$$

and

$$B[M, r] = \{x \in X \mid dist(x, M) \leq r\}.$$

Then

$$\begin{aligned} H(M, N) &= \inf\{r > 0 \mid M \subseteq B(N, r) \text{ and } N \subseteq B(M, r)\} \\ &= \inf\{r > 0 \mid M \subseteq B[N, r] \text{ and } N \subseteq B[M, r]\}. \end{aligned}$$

It follows that

$$H(\{x_0\}, \{x_1, x_2, \ldots, x_m\}) = \max_{i=1}^{m} \|x_0 - x_i\|,$$

$$H(\{x_0\}, A) < r \Rightarrow A \subseteq B(x_0, r) = \{x \in X \mid \|x - x_0\| < r\}$$

and

$$H(\{x_0\}, A) \leq r \Rightarrow A \subseteq B[x_0, r] = \{x \in X \mid \|x - x_0\| \leq r\}.$$

We shall say that the system of functions $(T^1, T^2, \ldots, T^h)$ forms an *iterated function system* (like $(T_n^1, T_n^2, \ldots, T_n^h)$).

On the complete metric space $\mathcal{K}(X)$, we can define the Hutchinson contractions (see [16,1]) $F : \mathcal{K}(X) \to \mathcal{K}(X)$ and $F_n : \mathcal{K}(X) \to \mathcal{K}(X)$, via

$$F(B) = \bigcup_{i=1}^{h} T^i(B), \qquad F_n(B) = \bigcup_{i=1}^{h} T_n^i(B).$$

These contractions have contractions constants less or equal to r (see (2) and (4′)) according to the general theory.

The fixed point A for F (called the attractor of the iterated function system $(T^1, T^2, \ldots, T^h)$) is, generally speaking, a fractal. The attractor of F_n will be denoted by A_n. Our aim in the sequel will be to approximate A with a finite set and to "draw a picture" of the finite approximant.

Let us notice some preliminary facts concerning the attractors. For any contraction $V : \mathcal{K}(X) \to \mathcal{K}(X)$, having contraction constant $\alpha \in (0, 1)$, the attractor of V can be obtained as follows (general theory): (a) we take an arbitrary $B_0 \in \mathcal{K}(X)$; (b) we define the sequence $(B_n)_n$ via $B_n = V(B_{n-1})$, $n = 1, 2, \ldots$; (c) $\lim_n B_n = B =$ the attractor of V. For any natural n one has

$$H(B, B_n) \leq \frac{\alpha^n}{1 - \alpha} H(B_1, B_0) \tag{7}$$

and, consequently

$$H(B, B_0) \leq \frac{1}{1-\alpha} H(B_1, B_0). \tag{7'}$$

We shall work for $V = F$ or $V = F_n$ and $B_0 = \{0\}$. In this case it is seen that $V(B_0) = \{f^1, f^2, \ldots, f^h\}$, hence

$$H(B_1, B_0) = \max_{i=1}^{h} \left\| f^i - 0 \right\| = M$$

(see (3)).

It follows, using (7′), that one has

$$A \subseteq B\left[0, \frac{M}{1-r}\right] \tag{8}$$

which implies that for $x \in A$ one has

$$\|x\| \leq \frac{M}{1-r} \tag{9}$$

and, consequently,

$$diam(A) \leq \frac{2M}{1-r}.$$

It is known that, under special assumptions, $A_n \underset{n}{\rightarrow} A$. The following result follows this line, giving an estimation of the distance between A_n and A.

Theorem 1. *For all n, one has*

$$H(A, A_n) \leq \frac{M}{(1-r)^2} r_n \underset{n}{\rightarrow} 0. \tag{10}$$

Hence

$$A_n \underset{n}{\rightarrow} A.$$

Proof. One has

$$\begin{aligned} H(A_n, A) &= H(F_n(A_n), F(A)) \leq H(F_n(A_n), F_n(A)) + H(F_n(A), F(A)) \\ &\leq rH(A_n, A) + H(F_n(A), F(A)). \end{aligned}$$

It follows that

$$\begin{aligned} (1-r)H(A_n, A) &\leq H(F_n(A), F(A)) \\ &= H\left(\bigcup_{i=1}^{h} T_n^i(A), \bigcup_{i=1}^{h} T^i(A)\right) \leq \max_{i=1}^{h} H(T_n^i(A), T^i(A)), \end{aligned} \tag{11}$$

according to a property of H.

But, according to (5), (6) and (9), one has

$$\begin{aligned} d(T_n^i(A), T^i(A)) &= \sup_{u \in A} dist(T_n^i(u), T^i(A)) = \sup_{u \in A} \inf_{v \in A} \left\| T_n^i(u) - T^i(v) \right\| \\ &\leq \sup_{u \in A} \left\| T_n^i(u) - T^i(u) \right\| \leq \sup_{u \in A} r_n \|u\| \leq r_n \frac{M}{1-r} \end{aligned} \tag{12}$$

and the same idea gives

$$d(T^i(A), T_n^i(A)) \leq r_n \frac{M}{1-r}. \tag{13}$$

From (12) and (13), we obtain, for all $i = 1, 2, \ldots, h$,

$$H(T_n^i(A), T^i(A)) = \max(d(T_n^i(A), T^i(A)), d(T^i(A)), T_n^i(A)) \leq r_n \frac{M}{1-r}$$

and from (11) it follows that

$$(1-r)H(A_n, A) \le r_n \frac{M}{1-r},$$

and, finally,

$$H(A_n, A) \le \frac{M}{(1-r)^2} r_n. \quad \square$$

This has been the first step of our approximation (namely, we approximate A with a convenient A_n).

In the sequel we pass to the second (and final) step of our approximation. Namely, for a given n, we shall approximate A_n with a finite set. To this aim, we shall fix n, we shall put $F_n = V$ and we shall take $B_0 = \{0\}$. Forming the sequence $(B_p)_p$ given via $B_p = V(B_{p-1})$, we obtain the finite sets B_p and $B_p \underset{p}{\rightarrow} A_n$.

Theorem 2. *Using the previous notations for a given n, one has, for all natural p,*

$$H(A_n, B_p) \le \frac{M}{1-r} r^p. \tag{14}$$

Proof. According to the general theory, one has (see (7))

$$H(A_n, V^p(B_0)) = H(A_n, B_p) \le \frac{r^p}{1-r} H(V(B_0), B_0),$$

because the contraction constant of V is less than or equal to r.

We took $B_0 = \{0\}$. Then $B_1 = V(B_0) = \{f^1, f^2, \ldots, f^h\}$ and $H(V(B_0), B_0) = M$ as we have seen, and (14) is proved. $\square$

Remark 1. The set B_1 has h elements, the set B_2 has h^2 elements and, generally, the set B_p has h^p elements.

Conclusion of this part. For a given $\delta > 0$ one can construct a finite set B_p which approximates A such that

$$H(A, B_p) < \delta \tag{15}$$

as follows:

(a) Write $\delta = \delta_1 + \delta_2$, with $\delta_1 > 0, \delta_2 > 0$.

(b) Choose a convenient n such that

$$\frac{M}{(1-r)^2} r_n < \delta_1. \tag{16}$$

(c) For the number n found at (b), choose a number p such that one has

$$\frac{M}{1-r} r^p < \delta_2 \tag{17}$$

and writing $V = F_n$, construct effectively the finite set B_p.

(d) According to (10) and (14), together with (16) and (17), one has

$$\begin{aligned} H(A, B_p) &\le H(A, A_n) + H(A_n, B_p) \\ &\le \frac{M}{(1-r)^2} r_n + \frac{M}{1-r} r^p < \delta_1 + \delta_2 = \delta, \end{aligned}$$

and (15) is proved.

Remark 2. The intermediate set A_n appears only theoretically, generating the number n.

Remark 3. An alternative procedure of obtaining B_p is described in [14].

Numerical example. Take $[a, b] = [0, 1]$, $h = 2$, $K^1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, $K^2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, $f^1 : [0, 1] \rightarrow \mathbb{R}$, $f^2 : [0, 1] \rightarrow \mathbb{R}$ given by

$$K^1(x, y) = e^{xy};$$
$$K^2(x, y) = \cos xy;$$
$$f^1(x) = 1;$$
$$f^2(x) = -1.$$

Because $\|K_1\| = e$, $\|K_2\| = 1$, one has $C = e$.
Since $\left\|f^1\right\| = \left\|f^2\right\| = 1$, one has $M = 1$.
We can take $\varepsilon = 3 - e$ and $\lambda = \frac{1}{10}$.
So, we have $r = |\lambda|\,(b-a)(C+\varepsilon) = \frac{3}{10}$, hence $1 - r = \frac{7}{10}$.
It follows (see (9)) that $diam(A) \le \frac{20}{7} < 3$.
For an integer $n \ge 1$, we shall take P_n^1 and P_n^2 as follows

$$P_n^1(x,y) = 1 + \frac{xy}{1!} + \frac{(xy)^2}{2!} + \cdots + \frac{(xy)^{2n}}{(2n)!}$$

(because $e^{xy} = \sum_{p=0}^{\infty} \frac{(xy)^p}{p!}$) and

$$P_n^2(x,y) = 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} \cdots + (-1)^n \frac{(xy)^{2n}}{(2n)!}$$

(because $\cos xy = \sum_{p=0}^{\infty} (-1)^p \frac{(xy)^{2p}}{(2p)!}$).

In order to obtain ε_n and r_n (see (4) and (6)) one notices that for $i = 1, 2$, for all $n \ge 2$ and for all $(x,y) \in [0,1] \times [0,1]$, one has

$$\begin{aligned}\left|K^i(x,y) - P_n^i(x,y)\right| &\le \sum_{k=2n+1}^{\infty} \frac{(xy)^k}{k!}\\ &\le \sum_{k=2n+1}^{\infty} \frac{1}{k!} \le \frac{1}{(2n+1)!}\left(1 + \frac{1}{2n+2} + \frac{1}{(2n+2)^2} + \cdots\right)\\ &= \frac{1}{(2n+1)!} \cdot \frac{1}{1 - \frac{1}{2n+2}} = \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1},\end{aligned}$$

so one can take

$$\varepsilon_n = \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1} < 3 - e = \varepsilon.$$

It follows that

$$r_n \le |\lambda|\,(b-a)\varepsilon_n = \frac{1}{10} \cdot \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1}.$$

We shall work for $\delta = \frac{1}{100}$.
So, we want to have

$$H(A, B_p) < \frac{1}{100}.$$

Let us take $\delta_1 = \delta_2 = \frac{1}{200}$.
Relationship (16) is fulfilled if

$$\frac{10^2}{7^2} \cdot \frac{1}{10} \cdot \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1} < \frac{1}{200} \iff \frac{2n+2}{(2n+1)!(2n+1)} < \frac{49}{2000} \iff n \ge 2.$$

We can take $n = 2$, which means

$$P_n^1(x,y) = P_2^1(x,y) = 1 + \frac{xy}{1!} + \frac{(xy)^2}{2!} + \frac{(xy)^3}{3!} + \frac{(xy)^4}{4!}$$

and

$$P_n^2(x,y) = P_2^2(x,y) = 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!}.$$

So, we shall take $V = F_2$.
Relationship (17) means

$$\frac{10}{7} \cdot \frac{3^p}{10^p} < \frac{1}{200} \iff \frac{1}{7} \cdot \frac{3^p}{10^{p-3}} < \frac{1}{2} \iff 10^{p-3} > \frac{2}{7} 3^p \iff p \ge 5.$$

We can take $p = 5$.
So, we have $n = 2$, $p = 5$.
The approximating set for the fractal will have $2^5 = 32$ points.

4. Bidimensional projection

In order to make computation easier, throughout this section we shall work in the case $[a, b] = [0, 1]$. Write $L^2 = L^2(\mu)$, where μ is the Lebesgue measure on $[0, 1]$. The Banach space L^2 is equipped with the usual norm

$$\left\|\tilde{f}\right\|_2 = \left(\int f^2 \mathrm{d}\mu\right)^{\frac{1}{2}}.$$

We have the linear, injective and continuous map (embedding) $I : X \to L^2$, given by

$$I(f) = \tilde{f}$$

(notice that $\|I(f)\|_2 \leq \|f\|$).

The space L^2 contains the (closed) bidimensional subspace

$$Z = \{\tilde{f} \in L^2 \mid f(x) = ax + b, a, b \in \mathbb{R}\}$$

consisting of all (classes of) polynomials of degree less or equal to 1.

Let $P : L^2 \to Z$ be the orthogonal projection. For any $\tilde{f} \in L^2$, one has $P(\tilde{f}) = \tilde{g} \in Z$, where $g(x) = ax + b$ for some real a and b. One has

$$\begin{aligned}\left\|\tilde{f} - \tilde{g}\right\| &= \inf\left\{\left\|\tilde{f} - \tilde{u}\right\|_2 \mid \tilde{u} \in Z\right\} \\ &= \min\left\{\left\|\tilde{f} - \tilde{u}\right\|_2 \mid \tilde{u} \in Z\right\}.\end{aligned}$$

So, we have the linear and continuous map $P \circ I : X \to Z$. We are interested in the projection of the fractal A, so we are interested in $P(I(A))$. We shall approximate $P(I(A))$ with $P(I(B_p))$, because, due to the fact that $P \circ I$ is Lipschitz, we have the implication $B_p \to A \Rightarrow (P \circ I)(B_p) \to (P \circ I)(A)$. Namely, $H((P \circ I)(A), (P \circ I)(B_p)) \leq \|P \circ I\| \, H(A, B_p) \leq H(A, B_p)$.

Because the elements of B_p are polynomial functions, take a natural n and let us compute $(P \circ I)(u)$, where $u(x) = x^n$. We have $(P \circ I)(u) = \tilde{v}$, where $v(x) = ax + b$ and the real numbers a, b must be such that $\left\|\tilde{u} - \tilde{v}\right\|_2$ is minimum. This is equivalent to the fact that

$$\int_0^1 (x^n - ax - b)^2 \mathrm{d}x = \int_0^1 (x^{2n} + a^2x^2 + b^2 - 2ax^{n+1} - 2bx^n + 2abx)\mathrm{d}x$$

is minimum.

So, we must minimize the function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$\varphi(a, b) = \frac{1}{3}a^2 + ab + b^2 - \frac{2}{n+2}a - \frac{2}{n+1}b + \frac{1}{2n+1}.$$

The minimum point will be the solution of the system

$$\begin{cases} \dfrac{\delta\varphi}{\delta a}(a, b) = \dfrac{2}{3}a + b - \dfrac{2}{n+2} = 0 \\ \dfrac{\delta\varphi}{\delta b}(a, b) = a + 2b - \dfrac{2}{n+1} = 0, \end{cases}$$

namely

$$a = \frac{6n}{(n+1)(n+2)}, \qquad b = \frac{-2n+2}{(n+1)(n+2)}.$$

So, informally, one has

$$(P \circ I)(x^n) = \frac{6n}{(n+1)(n+2)}x + \frac{-2n+2}{(n+1)(n+2)}$$

and, for a general polynomial function

$$\left(P \circ I\right)(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \left(\sum_{k=0}^{n} \frac{6k}{(k+1)(k+2)}a_k\right)x + \left(\sum_{k=0}^{n} \frac{-2k+2}{(k+1)(k+2)}a_k\right). \tag{18}$$

In order to "draw the picture" of the projection of B_p, we shall proceed as follows:

(a) We shall effectively construct B_p (namely $n = 2$, $p = 5$, which gives 32 points for B_p; see the numerical example).

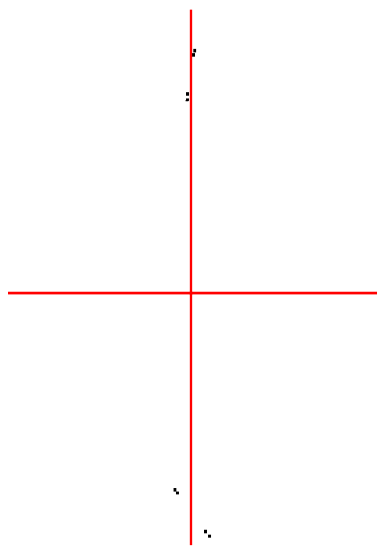
(b) The 32 elements of B_p are polynomials. Each polynomial $u \in B_p$ will be projected onto a polynomial of the form $ax + b$ according to (18) and we shall identify $ax + b \equiv (a, b)$. Thus, one obtains 32 points in the Cartesian plane, which depict the projection of B_p and an approximate image of the projection of A.

5. Concrete results: Bidimensional finite approximation of the fractal

Using a Java program, the authors obtained the following 32 points (a, b) in the Cartesian plane:

a	b
0.08304457859572774	1.1107585050156217
0.08266313448530387	1.1102984928879296
0.08016730105296681	1.1072865544959738
0.079902906929636	1.1069674319504388
0.06181902581776706	1.0849548057802325
0.06156572215602225	1.0846456756949472
0.059844065895693195	1.0825454553578604
0.05965993171301646	1.0823208565995743
−0.06362714552188674	0.9142524282016085
−0.0638550629798198	0.9139306050144175
−0.06535197983681224	0.9118179132178063
−0.06551070803674693	0.9115939168795805
−0.07688243358677374	0.8956350231963166
−0.0770439319637552	0.8954087504478606
−0.07813916642196006	0.8938738332662388
−0.07825623829837865	0.8937097519935937
−0.018765102776208124	−0.8817425488527297
−0.018676763532994384	−0.8822380380815367
−0.01809884094674398	−0.8854820631131635
−0.018037621673061863	−0.8858257703660861
−0.013858743937846528	−0.9095158858766432
−0.013800245780316507	−0.9098484953936015
−0.01340260538436051	−0.9121083200919652
−0.013360075977001439	−0.9123499893332517
0.014339845705302705	−1.0914751373621152
0.014390514788600746	−1.0918171377533887
0.014723345802773288	−1.0940623720514837
0.014758639386260548	−1.0943004233808342
0.017291384889613442	−1.1112683061588968
0.01732737192612747	−1.1115089168880086
0.017571406339598843	−1.1131410607556897
0.017597491170256302	−1.1133155345406052

A second program draws in the plane the 32 result points.
These points are grouped in four "clouds" (only the last one will appear here).
We get the following picture:



Unfortunately, the four "clouds" of points are too scattered, by comparison with their size. We draw the last one (corresponding to the last 8 pairs of coordinates) using a large scale.

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